# ON THE QUANTUM INVARIANT FOR THE BRIESKORN HOMOLOGY SPHERES

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ABSTRACT. We study an exact asymptotic behavior of the Witten–Reshetikhin–Turaev SU(2) invariant for the Brieskorn homology spheres $\Sigma(p_1, p_2, p_3)$ by use of properties of the modular form following a method proposed by R. Lawrence and D. Zagier. Key observation is that the invariant coincides with a limiting value of the Eichler integral of the modular form with weight $3/2$. We show that the Casson invariant is related to the number of the Eichler integrals which do not vanish in a limit $\tau \to N \in \mathbb{Z}$. Correspondingly there is a one-to-one correspondence between the non-vanishing Eichler integrals and the irreducible representation of the fundamental group, and the Chern–Simons invariant is given from the Eichler integral in this limit. It is also shown that the Ohtsuki invariant follows from a nearly modular property of the Eichler integral, and we give an explicit form in terms of the $L$-function.

## 1. INTRODUCTION

The quantum invariant for the 3-manifold $\mathcal{M}$ was introduced as a path integral on $\mathcal{M}$ by Witten [43]; as the $SU(2)$ invariant we have

$$Z_k(\mathcal{M}) = \int \exp\Bigl(2\,\pi\,\mathrm{i}\,k\;\mathrm{CS}(A)\Bigr)\,\mathcal{D}A \tag{1.1}$$

where $k \in \mathbb{Z}$, and $\mathrm{CS}(A)$ is the Chern–Simons functional defined by

$$\mathrm{CS}(A) = \frac{1}{8\,\pi^2} \int\limits_{\mathcal{M}} \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}\,A \wedge A \wedge A\right) \tag{1.2}$$

Since this work, studies of the quantum invariants of the 3-manifolds have been extensively developed, and a construction of the 3-manifold invariant was reformulated combinatorially and rigorously in Refs. 15, 33 using a surgery description of $\mathcal{M}$ and the colored Jones polynomial defined in Ref. 14.

By applying a stationary phase approximation, an asymptotic behavior of the Witten invariant in large $k \to \infty$ is expected to be [6, 43] (see also Ref. 2)

$$Z_k(\mathcal{M}) \sim \frac{1}{2}\,\mathrm{e}^{-\frac{3}{4}\pi\mathrm{i}} \sum_{\alpha} \sqrt{T_\alpha(\mathcal{M})}\,\mathrm{e}^{-2\pi\mathrm{i}I_\alpha/4}\,\mathrm{e}^{2\pi\mathrm{i}(k+2)\,\mathrm{CS}(A)} \tag{1.3}$$

Here the sum runs over a flat connection $\alpha$, and $T_\alpha$ and $I_\alpha$ are the Reidemeister torsion and the spectral flow defined modulo $8$ respectively.

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In this article we consider the Brieskorn homology spheres $\Sigma(p_1,p_2,p_3)$ where $p_i$ are pairwise coprime positive integers. This is the intersection of the singular complex surface

$$z_1^{\,p_1} + z_2^{\,p_2} + z_3^{\,p_3} = 0$$

in complex three-space with the unit five-sphere $|z_1|^2+|z_2|^2+|z_3|^2=1$. The manifold $\Sigma(2,3,5)$ is the Poincaré homology sphere. These manifolds have a rational surgery description as in Fig. 1, and the fundamental group has the presentation

$$\pi_1\bigl(\Sigma(p_1,p_2,p_3)\bigr) = \Bigl\langle x_1,x_2,x_3,h \bigm| h \text{ center}, x_k^{p_k} = h^{-q_k} \text{ for } k=1,2,3, x_1\,x_2\,x_3=1\Bigr\rangle \tag{1.4}$$

where $q_k \in \mathbb{Z}$ such that

$$P \sum_{k=1}^{3} \frac{q_k}{p_k} = 1 \tag{1.5}$$

Here and hereafter we use

$$P = P(p_1,p_2,p_3) = p_1\,p_2\,p_3 \tag{1.6}$$

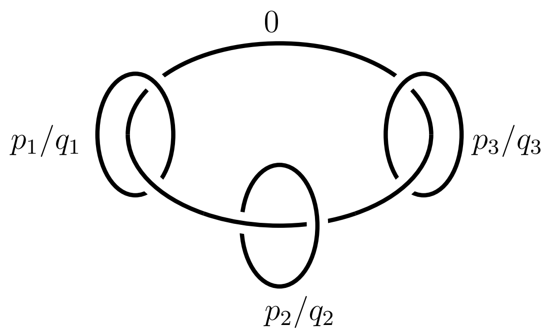


Figure 1: Rational surgery description of the Brieskorn homology sphere $\Sigma(p_1,p_2,p_3)$

Asymptotic behavior of the quantum invariant for the homology spheres was studied in Refs. 19, 20, 22, 34–39. Our purpose here is to reformulate these results number theoretically by use of properties of the modular form following a method of Lawrence and Zagier [23]. A key observation is a fact that the Witten–Reshetikhin–Turaev (WRT) invariant for the Brieskorn homology spheres is regarded as a limit value of the Eichler integral of the modular form with weight $3/2$ as was suggested in Ref. 23. Using a nearly modular property of the Eichler integral, we can derive an exact asymptotic behavior of the WRT invariant. Correspondingly, we can find an interpretation for topological invariants such as the Chern–Simons invariant, the Casson invariant [42], the Ohtsuki invariant [27–30], and the Ray–Singer–Reidemeister torsion from the viewpoint of the modular form.

This paper is organized as follows. In section 2 we construct the WRT invariant for the Brieskorn homology spheres $\Sigma(p_1,p_2,p_3)$ following Ref. 22. We use a surgery description of the 3-manifold, and apply a formula in Ref. 13. In section 3 we introduce the modular form with weight $3/2$. This

gives a $(p_1-1)\,(p_2-1)\,(p_3-1)/4$-dimensional representation of the modular group $PSL(2;\mathbb{Z})$. We consider the Eichler integral thereof in section 4. We study a limiting value of the Eichler integral at $\tau\in\mathbb{Q}$, and show that the number of the Eichler integrals which have non-zero value in a limit $\tau\to N\in\mathbb{Z}$ is related to the Casson invariant of the Brieskorn homology spheres. It will be discussed that we have a one-to-one correspondence with the irreducible representation of the fundamental group, and that the Chern–Simons invariant is given from this limiting value of the Eichler integral. We further give a nearly modular property of the Eichler integral. In section 5 we reveal a key identity that the WRT invariant for the Brieskorn spheres coincides with a limiting value of the Eichler integral at $\tau\to 1/N$. This was suggested in Ref. 23 where a correspondence was proved only for a case of the Poincaré homology sphere. Combining with results in section 4 we obtain an exact asymptotic expansion of the WRT invariant which is constituted from two terms; one is a sum of dominating exponential term which gives the Chern–Simons term, and another is a "tail" part which may be regarded as a contribution from a trivial connection. In section 6 we prove that the $\mathbf{S}$-matrix of the modular transformation gives both the Reidemeister torsion and the spectral flow. We show in section 7 that a tail part gives the Ohtsuki invariant. Explicitly computed is the $n$-th Ohtsuki invariant in terms of the $L$-function. The last section is devoted to concluding remarks.

## 2. The Witten–Reshetikhin–Turaev Invariant for the Brieskorn Homology Spheres

We introduce the Reshetikhin–Turaev invariant $\tau_N(\mathcal{M})$ [33] for $N\in\mathbb{Z}_+$. This is related to the Witten invariant $Z_k(\mathcal{M})$ defined in eq. (1.1) as

$$Z_k(\mathcal{M})=\frac{\tau_{k+2}(\mathcal{M})}{\tau_{k+2}(S^2\times S^1)} \tag{2.1}$$

Here the invariant is normalized to be

$$\tau_N(S^3)=1$$

and we have

$$\tau_N(S^2\times S^1)=\sqrt{\frac{N}{2}}\,\frac{1}{\sin(\pi/N)}$$

When the 3-manifold $\mathcal{M}$ is constructed by the rational surgeries $p_j/q_j$ on the $j$-th component of $n$-component link $\mathcal{L}$, it was shown [13, 33] that the invariant $\tau_N(\mathcal{M})$ is given by

$$\tau_N(\mathcal{M})=\mathrm{e}^{\frac{\pi \mathrm{i}}{4}\frac{N-2}{N}\left(\sum_{j=1}^{n}\Phi(U^{(p_j,q_j)})-3\,\mathrm{sign}(\mathbf{L})\right)}\sum_{k_1,\dots,k_n=1}^{N-1}J_{k_1,\dots,k_n}(\mathcal{L})\prod_{j=1}^{n}\rho(U^{(p_j,q_j)})_{k_j,1} \tag{2.2}$$

Here the surgery $p_j/q_j$ is described by an $SL(2;\mathbb{Z})$ matrix

$$U^{(p_j,q_j)} = \begin{pmatrix} p_j & r_j \\ q_j & s_j \end{pmatrix}$$

and $\Phi(U)$ is the Rademacher $\Phi$-function defined by (see, *e.g.*, Ref. 31)

$$\Phi\left[\begin{pmatrix} p & r \\ q & s \end{pmatrix}\right] = \begin{cases} \dfrac{p+s}{q} - 12\, s(p,q) & \text{for } q \neq 0 \\ \dfrac{r}{s} & \text{for } q = 0 \end{cases} \tag{2.3}$$

where $s(b,a)$ denotes the Dedekind sum (see, *e.g.*, Ref. 32, and also Ref. 16)

$$s(b,a) = \operatorname{sign}(a) \sum_{k=1}^{|a|-1} \left(\left(\frac{k}{a}\right)\right) \cdot \left(\left(\frac{k\, b}{a}\right)\right) \tag{2.4}$$

with

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

and $\lfloor x \rfloor$ is the greatest integer not exceeding $x$. It is known that the Dedekind sum is rewritten as

$$s(b,a) = \frac{1}{4\,a} \sum_{k=1}^{a-1} \cot\left(\frac{k}{a}\,\pi\right) \cot\left(\frac{k\,b}{a}\,\pi\right)$$

An $n \times n$ matrix $\mathbf{L}$ is a linking matrix $\mathbf{L}_{j,k} = \mathrm{lk}_{j,k} + p_j/q_j \cdot \delta_{j,k}$, and $\operatorname{sign}(\mathbf{L})$ is a signature of $\mathbf{L}$, *i.e.*, the difference between the number of positive and negative eigenvalues of $\mathbf{L}$. The polynomial $J_{k_1,\dots,k_n}(\mathcal{L})$ is the colored Jone polynomial for link $\mathcal{L}$ with the color $k_j$ for the $j$-th component link, and $\rho(U^{(p,q)})$ is a representation $\rho$ of $PSL(2;\mathbb{Z})$;

$$\begin{aligned} &\rho(U^{(p,q)})_{a,b} \\ &= -\mathrm{i} \frac{\operatorname{sign}(q)}{\sqrt{2\,N\,|q|}}\, \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}\,\Phi(U^{(p,q)})}\, \mathrm{e}^{\frac{\pi \mathrm{i}}{2Nq} s b^2} \sum_{\substack{\gamma \mod 2Nq \\ \gamma = a \mod 2N}} \mathrm{e}^{\frac{\pi \mathrm{i}}{2Nq} p \gamma^2} \left( \mathrm{e}^{\frac{\pi \mathrm{i}}{Nq}\gamma b} - \mathrm{e}^{-\frac{\pi \mathrm{i}}{Nq}\gamma b} \right) \end{aligned} \tag{2.5}$$

for $1 \leq a, b \leq N-1$ [13], and we have

$$\begin{aligned} \rho(S)_{a,b} &= \sqrt{\frac{2}{N}}\, \sin\left(\frac{a\,b\,\pi}{N}\right) \\ \rho(T)_{a,b} &= \mathrm{e}^{\frac{\pi \mathrm{i}}{2N} a^2 - \frac{\pi \mathrm{i}}{4}}\, \delta_{a,b} \end{aligned} \tag{2.6}$$

with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

satisfying

$$S^2 = (S\,T)^3 = 1$$

We should note [32] that the Dedekind sum satisfies

$$s(-b,a) = -s(b,a) \tag{2.8}$$

$$s(b,a) = s(b',a) \quad \text{for } b\,b' \equiv 1 \pmod{a} \tag{2.9}$$

and that the Rademacher $\Phi$-function fulfills

$$\Phi(S\,U^{(p,q)}) = \Phi(U^{(p,q)}) - 3\ \mathrm{sign}(p\,q) \tag{2.10}$$

**Proposition 1** ( [22]). *The WRT invariant for the Brieskorn homology spheres is given by*

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}(\frac{\phi}{4}-\frac{1}{2})}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}-1\right)\tau_N\big(\Sigma(p_1,p_2,p_3)\big)$$

$$= \frac{\mathrm{e}^{\pi\mathrm{i}/4}}{2\sqrt{2\,P\,N}} \sum_{\substack{n=0\\ N\nmid n}}^{2\,P\,N-1} \mathrm{e}^{-\frac{1}{2PN}n^2\pi\mathrm{i}} \frac{\prod_{j=1}^{3}\left(\mathrm{e}^{\frac{n}{Np_j}\pi\mathrm{i}}-\mathrm{e}^{-\frac{n}{Np_j}\pi\mathrm{i}}\right)}{\mathrm{e}^{\frac{n}{N}\pi\mathrm{i}}-\mathrm{e}^{-\frac{n}{N}\pi\mathrm{i}}} \tag{2.11}$$

*where*

$$\phi = \phi(p_1,p_2,p_3) = 3 - \frac{1}{P} + 12\left(s(p_2\,p_3,p_1)+s(p_1\,p_3,p_2)+s(p_1\,p_2,p_3)\right) \tag{2.12}$$

*Proof.* This was proved in Refs. 22, 38 for a general $n$-fibered manifold, but we give a proof here again for completion.

The Jones polynomial for a link $\mathcal{L}$ depicted in Fig. 1 is given by

$$J_{k_0,k_1,k_2,k_3}(\mathcal{L}) = \frac{1}{\sin(\pi/N)} \cdot \frac{\prod_{j=1}^{3}\sin\left(k_0\,k_j\,\pi/N\right)}{\sin^2\left(k_0\,\pi/N\right)}$$

where $k_0$ is a color of an unknotted component whose linking number with other components is $1$, and $k_j$ (for $j=1,2,3$) denotes a color of a component of link $\mathcal{L}$ which is to be $p_j/q_j$-surgery. With this setting we have

$$\mathrm{sign}(\mathbf{L}) = \sum_{j=1}^{3}\mathrm{sign}\left(\frac{q_j}{p_j}\right) - 1$$

From (2.2) we get the quantum invariant as

$$\left(\mathrm{e}^{\pi\mathrm{i}/N}-\mathrm{e}^{-\pi\mathrm{i}/N}\right)\cdot\tau_N\big(\Sigma(p_1,p_2,p_3)\big)$$

$$= \mathrm{e}^{\frac{\pi\mathrm{i}}{4}\frac{N-2}{N}\left(3+\sum_{j=1}^{3}\Phi(SU^{(p_j,q_j)})\right)} \sum_{k_0=1}^{N-1} \frac{\rho(S)_{k_0,1}}{\left(\mathrm{e}^{\frac{\pi\mathrm{i}}{N}k_0}-\mathrm{e}^{-\frac{\pi\mathrm{i}}{N}k_0}\right)^2}$$

$$\times \prod_{j=1}^{3}\sum_{k_j=1}^{N-1}\rho\big(U^{(p_j,q_j)}\big)_{k_j,1}\left(\mathrm{e}^{\frac{\pi\mathrm{i}}{N}k_0k_j}-\mathrm{e}^{-\frac{\pi\mathrm{i}}{N}k_0k_j}\right)$$

In this expression we have by definition

$$\rho(S)_{k_0,1} = \frac{-\mathrm{i}}{\sqrt{2\,N}}\left(\mathrm{e}^{\frac{\pi\mathrm{i}}{N}k_0}-\mathrm{e}^{-\frac{\pi\mathrm{i}}{N}k_0}\right)$$

and for $\ell \in \mathbb{Z}$ we have

$$
\begin{aligned}
&\sum_{k=1}^{N-1} \rho(U^{(p,q)})_{k,1} \left(\mathrm{e}^{\frac{\pi \mathrm{i}}{N}\ell k} - \mathrm{e}^{-\frac{\pi \mathrm{i}}{N}\ell k}\right) \\
&= -\mathrm{i}\,\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}\Phi(U^{(p,q)})+\frac{\pi \mathrm{i} s}{2Nq}} \frac{\operatorname{sign}(q)}{\sqrt{2\,N\,|q|}} \sum_{k=1}^{N-1} \sum_{\substack{\gamma \mod 2Nq \\ \gamma \equiv k \mod 2N}} \mathrm{e}^{\frac{\pi \mathrm{i} p}{2Nq}\gamma^2} \left(\mathrm{e}^{\frac{\pi \mathrm{i}}{Nq}\gamma} - \mathrm{e}^{-\frac{\pi \mathrm{i}}{Nq}\gamma}\right) \left(\mathrm{e}^{\frac{\pi \mathrm{i}}{N}\ell\gamma} - \mathrm{e}^{-\frac{\pi \mathrm{i}}{N}\ell\gamma}\right) \\
&= -\mathrm{i}\,\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}\Phi(U^{(p,q)})+\frac{\pi \mathrm{i} s}{2Nq}} \frac{\operatorname{sign}(q)}{\sqrt{2\,N\,|q|}} \sum_{\gamma \mod 2Nq} \mathrm{e}^{\frac{\pi \mathrm{i}}{2Nq}p\gamma^2} \left(\mathrm{e}^{2\pi \mathrm{i}\frac{q\ell+1}{2Nq}\gamma} - \mathrm{e}^{2\pi \mathrm{i}\frac{q\ell-1}{2Nq}\gamma}\right) \\
&= -\frac{\operatorname{sign}(p)}{\sqrt{|p|}}\,\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}\Phi(SU^{(p,q)})+\frac{\pi \mathrm{i} s}{2Nq}} \sum_{n \mod p} \left(\mathrm{e}^{-\frac{\pi \mathrm{i}}{2pqN}(2Nqn+q\ell+1)^2} - \mathrm{e}^{-\frac{\pi \mathrm{i}}{2pqN}(2Nqn+q\ell-1)^2}\right)
\end{aligned}
$$

Here in the first equality we have used $\gamma = k \pmod{2\,N}$, and applied a symmetry under $\gamma \to 2\,N\,q - \gamma$ in the second equality. In the last equality, we have used an identity $\mathrm{e}^{\frac{\pi \mathrm{i}}{2}(1-\operatorname{sign}(p))} = \operatorname{sign}(p)$, and the Gauss sum reciprocity formula [13]

$$
\sum_{n \mod N} \mathrm{e}^{\frac{\pi \mathrm{i}}{N}Mn^2+2\pi \mathrm{i} kn} = \sqrt{\left|\frac{N}{M}\right|}\,\mathrm{e}^{\frac{\pi \mathrm{i}}{4}\operatorname{sign}(NM)} \sum_{n \mod M} \mathrm{e}^{-\frac{\pi \mathrm{i}}{M}N(n+k)^2} \tag{2.13}
$$

where $N, M \in \mathbb{Z}$ with $N\,k \in \mathbb{Z}$ and $N\,M$ being even.

A combination of these results reduces to

$$
\begin{aligned}
&\left(\mathrm{e}^{\pi \mathrm{i}/N} - \mathrm{e}^{-\pi \mathrm{i}/N}\right) \cdot \tau_N\left(\Sigma(p_1,p_2,p_3)\right) \\
&\qquad = \frac{\mathrm{i}\,\operatorname{sign}(P)}{\sqrt{2\,|P|\,N}}\,\mathrm{e}^{\frac{3}{4}\pi \mathrm{i}+\frac{\pi \mathrm{i}}{2N}\sum_j \frac{1}{p_j q_j}-\frac{\pi \mathrm{i}}{2N}\phi} \sum_{k_0=1}^{N-1} \sum_{n_j \mod p_j} \frac{1}{\mathrm{e}^{\frac{\pi \mathrm{i}}{N}k_0} - \mathrm{e}^{-\frac{\pi \mathrm{i}}{N}k_0}} \\
&\qquad\qquad \times \prod_{j=1}^{3} \left(\mathrm{e}^{-\frac{\pi \mathrm{i}}{2Np_jq_j}(2Nq_jn_j+k_0q_j+1)^2} - \mathrm{e}^{-\frac{\pi \mathrm{i}}{2Np_jq_j}(2Nq_jn_j+k_0q_j-1)^2}\right)
\end{aligned}
$$

The summand is invariant under (i) $k_0 \to k_0 + 2\,N$ and $n_j \to n_j - 1$, (ii) $n_j \to n_j + p_j$. Using this symmetry and recalling that $p_j$ are pairwise coprime integers, the sum, $\sum_{k_0=1}^{N-1}\sum_{n_j \mod p_j}$, is transformed into a sum, $\sum_{\substack{k_0=a+2Nn \\ 1\leq a\leq N-1 \\ 0\leq n\leq P-1}}$, with setting all $n_j = 0$. As a result, we find

$$
\begin{aligned}
&\left(\mathrm{e}^{\pi \mathrm{i}/N} - \mathrm{e}^{-\pi \mathrm{i}/N}\right) \cdot \tau_N\left(\Sigma(p_1,p_2,p_3)\right) \\
&\qquad = -\frac{\mathrm{i}}{\sqrt{2\,P\,N}}\,\mathrm{e}^{\frac{3}{4}\pi \mathrm{i}-\frac{\pi \mathrm{i}}{2N}\phi} \sum_{\substack{k_0=a+2Nn \\ 1\leq a\leq N-1 \\ 0\leq n\leq P-1}} \mathrm{e}^{-\frac{\pi \mathrm{i}}{2NP}{k_0}^2} \frac{\prod_{j=1}^{3}\left(\mathrm{e}^{\frac{\pi \mathrm{i}}{Np_j}k_0} - \mathrm{e}^{-\frac{\pi \mathrm{i}}{Np_j}k_0}\right)}{\mathrm{e}^{\frac{\pi \mathrm{i}}{N}k_0} - \mathrm{e}^{-\frac{\pi \mathrm{i}}{N}k_0}}
\end{aligned}
$$

Setting $k_0 \to 2\,P\,N - k_0$, we obtain a statement of the proposition. $\square$

## 3. Modular Forms

We define the odd periodic function $\chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n)$ with modulus $2\,P$ by

$$\chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n) = \begin{cases} 1 & \text{for } n = P\left(1+\sum_{j=1}^{3}\varepsilon_j\,\frac{\ell_j}{p_j}\right) \mod 2\,P \text{ where } \varepsilon_1\varepsilon_2\varepsilon_3=-1 \\ -1 & \text{for } n = P\left(1+\sum_{j=1}^{3}\varepsilon_j\,\frac{\ell_j}{p_j}\right) \mod 2\,P \text{ where } \varepsilon_1\varepsilon_2\varepsilon_3=1 \\ 0 & \text{others} \end{cases} \tag{3.1}$$

Here $P = p_1\,p_2\,p_3$ with pairwise coprime positive integers $p_j$, and we mean $\varepsilon_j = \pm 1$. Integers $\ell_j$ are

$$1 \leq \ell_j \leq p_j - 1 \tag{3.2}$$

There exists a symmetry of the periodic function

$$\begin{aligned} \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n) &= \chi_{2P}^{(p_1-\ell_1,p_2-\ell_2,\ell_3)}(n) \\ &= \chi_{2P}^{(p_1-\ell_1,\ell_2,p_3-\ell_3)}(n) = \chi_{2P}^{(\ell_1,p_2-\ell_2,p_3-\ell_3)}(n) \end{aligned} \tag{3.3}$$

With this periodic function, we define the function $\Phi_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau)$ for $\tau$ in the upper half plane, $\tau \in \mathbb{H}$, by

$$\Phi_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau) = \frac{1}{2}\sum_{n\in\mathbb{Z}} n\,\chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n)\,q^{\frac{n^2}{4P}} \tag{3.4}$$

where as usual

$$q = \exp(2\,\pi\,\mathrm{i}\,\tau)$$

Eq. (3.3) makes the number of the independent functions $\Phi_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau)$ to be

$$D = D(p_1,p_2,p_3) = \frac{1}{4}\,(p_1-1)\,(p_2-1)\,(p_3-1) \tag{3.5}$$

**Proposition 2.** *The function $\Phi_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau)$ is a modular form with weight $3/2$. Namely under the $S$- and $T$-transformations* (2.7) *we have*

$$\Phi_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau) = \left(\frac{\mathrm{i}}{\tau}\right)^{3/2}\sum_{\ell_1',\ell_2',\ell_3'}\mathbf{S}_{\ell_1,\ell_2,\ell_3}^{\ell_1',\ell_2',\ell_3'}\,\Phi_{\boldsymbol{p}}^{(\ell_1',\ell_2',\ell_3')}(-1/\tau) \tag{3.6}$$

$$\Phi_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau+1) = \mathbf{T}^{\ell_1,\ell_2,\ell_3}\Phi_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau) \tag{3.7}$$

*where the sum runs over $D(p_1, p_2, p_3)$ distinct triples. A $D \times D$ matrix $\mathbf{S}$ and diagonal matrix $\mathbf{T}$ are respectively given by*

$$\mathbf{S}_{\ell_1,\ell_2,\ell_3}^{\ell_1',\ell_2',\ell_3'} = \sqrt{\frac{32}{P}}\,(-1)^{1+P+P\sum_{j=1}^{3}\frac{\ell_j+\ell_j'}{p_j}+(\boldsymbol{\ell}\times\boldsymbol{\ell}')\cdot\boldsymbol{p}} \prod_{j=1}^{3} \sin\left(P\,\frac{\ell_j\,\ell_j'}{p_j^2}\,\pi\right) \tag{3.8}$$

$$\mathbf{T}^{\ell_1,\ell_2,\ell_3} = \exp\left(\frac{\pi\,\mathrm{i}}{2}\,P\left(1+\sum_{j=1}^{3}\frac{\ell_j}{p_j}\right)^2\right) \tag{3.9}$$

We omit a proof as it is tedious but straightforward. We only need the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}tn}\,f(t)\,\mathrm{d}t \tag{3.10}$$

## 4. The Eichler Integral and the Chern–Simons Invariant

The Eichler integral was originally defined as a $k-1$-fold integration of a modular form with integral weight $k \in \mathbb{Z}_{\geq 2}$ (see, *e.g.*, Ref. 18). Following Refs. 23, 44 (see also Refs. 9–11), we define the Eichler integral of the modular form $\Phi_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau)$ with *half-integral* weight $3/2$ by

$$\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau) = \sum_{n=0}^{\infty} \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n)\,q^{\frac{n^2}{4P}} \tag{4.1}$$

We should remark that there are $D(p_1, p_2, p_3)$ independent Eichler integrals due to the symmetry (3.3).

**Proposition 3.** *The function $\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau)$ has a limiting value in $\tau \to 1/N$ for $N \in \mathbb{Z}$ as*

$$\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(1/N) = \sum_{n=0}^{PN} \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n)\,\left(1-\frac{n}{P\,N}\right)\,\mathrm{e}^{\frac{1}{2PN}n^2\pi\mathrm{i}} \tag{4.2}$$

*We also have*

$$\begin{aligned}\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(N) &= -\frac{1}{2\,P}\,\left(\sum_{n=1}^{2P} n\,\chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n)\right)\,\mathrm{e}^{\frac{\pi\mathrm{i}}{2}PN\left(1+\sum_j\frac{\ell_j}{p_j}\right)^2} \\ &= -\frac{1}{2\,P}\,\left(\sum_{n=1}^{2P} n\,\chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n)\right)\,\left(\mathbf{T}^{\ell_1,\ell_2,\ell_3}\right)^N\end{aligned} \tag{4.3}$$

To prove this proposition, we use the following formula for asymptotic expansions (see Refs. 23, 44);

**Proposition 4.** *Let $C_f(n)$ be a periodic function with mean value $0$ and modulus $f$. Then we have an asymptotic expansion as $t \searrow 0$;*

$$\sum_{n=1}^{\infty} C_f(n)\, \mathrm{e}^{-nt} \simeq \sum_{k=0}^{\infty} L(-k, C_f)\, \frac{(-t)^k}{k!} \tag{4.4}$$

$$\sum_{n=1}^{\infty} C_f(n)\, \mathrm{e}^{-n^2 t} \simeq \sum_{k=0}^{\infty} L(-2\,k, C_f)\, \frac{(-t)^k}{k!} \tag{4.5}$$

*Here $L(k, C_f)$ is the Dirichlet $L$-function associated with $C_f(n)$, and is given by*

$$L(-k, C_f) = -\frac{f^k}{k+1} \sum_{n=1}^{f} C_f(n)\, B_{k+1}\left(\frac{n}{f}\right)$$

*where $B_n(x)$ is the $n$-th Bernoulli polynomial defined from*

$$\frac{t\,\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!}\, t^n$$

See, *e.g.*, Ref. 23 for a proof.

*Proof of Proposition 3.* We assume $M$ and $N$ are coprime integers, and $N > 0$. By definition, we have

$$\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}\left(\frac{M}{N} + \mathrm{i}\,\frac{y}{2\,\pi}\right) = \sum_{n=0}^{\infty} C_{2PN}^{(\ell_1,\ell_2,\ell_3)}(n)\, \mathrm{e}^{-\frac{y}{4P} n^2}$$

where $y > 0$ and

$$C_{2PN}^{(\ell_1,\ell_2,\ell_3)}(n) = \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n)\, \mathrm{e}^{\frac{M}{2PN} n^2 \pi \mathrm{i}}$$

We see that $C_{2PN}^{(\ell_1,\ell_2,\ell_3)}(n + 2\,P\,N) = C_{2PN}^{(\ell_1,\ell_2,\ell_3)}(n)$, and $C_{2PN}^{(\ell_1,\ell_2,\ell_3)}(2\,P\,N - n) = -C_{2PN}^{(\ell_1,\ell_2,\ell_3)}(n)$. Then we can apply Prop. 4, and we have an asymptotic expansion in $y \searrow 0$ as

$$\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}\left(\frac{M}{N} + \mathrm{i}\,\frac{y}{2\,\pi}\right) \simeq \sum_{k=0}^{\infty} \frac{L(-2\,k, C_{2PN}^{(\ell_1,\ell_2,\ell_3)})}{k!} \left(-\frac{y}{4P}\right)^k$$

which gives a limiting value as

$$\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(M/N) = L(0, C_{2PN}^{(\ell_1,\ell_2,\ell_3)}) \tag{4.6}$$

Using a fact that $\chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(2\,P - n) = -\chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n)$ and that an explicit form of the Bernoulli polynomial is $B_1(x) = x - \frac{1}{2}$, we get

$$\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(M/N) = \sum_{n=0}^{PN} \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n) \left(1 - \frac{n}{P\,N}\right) \mathrm{e}^{\frac{M}{2PN} n^2 \pi \mathrm{i}}$$

Eq. (4.2) directly follows from this formula. Eq. (4.3) can also be given from the above formula when we recall $\sum_{n=1}^{2P} \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n) = 0$. $\square$

We can see that, though we have $D(p_1, p_2, p_3)$ independent Eichler integrals, the limiting value $\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(N)$ at $N \in \mathbb{Z}$ computed in eq. (4.3) becomes identically zero for some triples $(\ell_1, \ell_2, \ell_3)$.

**Proposition 5.** *Let $\gamma(p_1, p_2, p_3)$ be the number of independent Eichler integrals such that $\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(N) \not\equiv 0$ for $N \in \mathbb{Z}$, namely*

$$\sum_{n=1}^{2P} n\, \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n) \neq 0 \tag{4.7}$$

*We then have*

$$\begin{aligned}\gamma(p_1, p_2, p_3) = s(p_1\, p_2, p_3) + s(p_2\, p_3, p_1) + s(p_1\, p_3, p_2) \\ + \frac{P}{12}\left(1 - \frac{1}{{p_1}^2} - \frac{1}{{p_2}^2} - \frac{1}{{p_3}^2}\right) - \frac{1}{12\, P} + \frac{1}{4}\end{aligned} \tag{4.8}$$

*where $s(b, a)$ is the Dedekind sum defined in eq.* (2.4).

*Proof.* For a sake of our brevity we set $A_k = \frac{\ell_i}{p_i} + \frac{\ell_j}{p_j} - \frac{\ell_k}{p_k}$ for $i \neq j \neq k \neq i$ and $i, j, k \in \{1, 2, 3\}$. As we have $0 < \frac{\ell_k}{p_k} < 1$ by definition, we have $0 < \sum_j \frac{\ell_j}{p_j} < 3$ and $-1 < A_k < 2$.

When $0 < \sum_j \frac{\ell_j}{p_j} < 1$, we have $0 < A_k + 1 < 2\left(1 - \frac{\ell_k}{p_k}\right) < 2$. Then in a domain $0 < n < 2\,P$, the periodic function $\chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n)$ defined in eq. (3.1) takes a value $1$ when $n = P\left(1 - \sum_j \frac{\ell_j}{p_j}\right), P(1 + A_k)$, while it is $-1$ when $n = P\left(1 + \sum_j \frac{\ell_j}{p_j}\right), P\ (1 - A_k)$. Then we find $\sum_{n=1}^{2P} n\chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n) = 0$, and it is inconsistent with eq. (4.7).

We thus have a condition $1 < \sum_j \frac{\ell_j}{p_j} < 3$ to fulfill eq. (4.7), because it is impossible to have $\sum_j \frac{\ell_j}{p_j} \in \mathbb{Z}$. Under this condition there are two possibilities for a condition of $A_k$; (i) $-1 < A_k < 1$ for all $k$, or (ii) $-1 < A_k < 1$ for two $k$'s and $1 < A_i < 2$ for another $i$. By the same computation we can check

$$\sum_{n=1}^{2P} n\, \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n) = 4\,P \tag{4.9}$$

for the former case, while we have $\sum_{n=1}^{2P} n\chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n) = 0$ for the latter. To conclude, a condition (4.7) is fulfilled when the triple of integers satisfies

$$\begin{aligned}&1 < \frac{\ell_1}{p_1} + \frac{\ell_2}{p_2} + \frac{\ell_3}{p_3} < 3 \qquad &-1 < \frac{\ell_1}{p_1} + \frac{\ell_2}{p_2} - \frac{\ell_3}{p_3} < 1 \\ &-1 < \frac{\ell_1}{p_1} - \frac{\ell_2}{p_2} + \frac{\ell_3}{p_3} < 1 \qquad &-1 < -\frac{\ell_1}{p_1} + \frac{\ell_2}{p_2} + \frac{\ell_3}{p_3} < 1\end{aligned} \tag{4.10}$$

This constraint is depicted as the number of the integral lattice points of an interior of the tetrahedron (see Fig. 2).

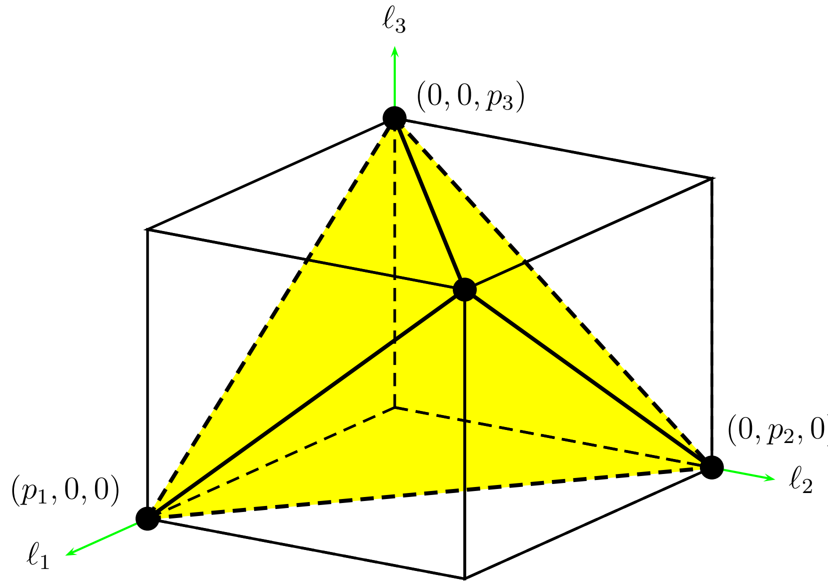

Figure 2: The number of the integral lattice points in the tetrahedron coincides with $\gamma(p_1,p_2,p_3)$.

Let $N_3(p_1,p_2,p_3)$ be the number of the integral lattice points $(\ell_1,\ell_2,\ell_3)$ such that $0<\ell_k<p_k$ and $0<\sum_j \frac{\ell_j}{p_j}<1$. Then by the symmetry of the triple,

$$\begin{aligned} &(\ell_1,\ell_2,\ell_3) && (\ell_1,p_2-\ell_2,p_3-\ell_3) \\ &(p_1-\ell_1,\ell_2,p_3-\ell_3) && (p_1-\ell_1,p_2-\ell_2,\ell_3) \end{aligned} \tag{4.11}$$

we have

$$\gamma(p_1,p_2,p_3)=D(p_1,p_2,p_3)-N_3(p_1,p_2,p_3)$$

It is well known that $N_3(p_1,p_2,p_3)$ was computed by Mordell [26] (see also Ref. 32), and by substituting this result, we get eq. (4.8). □

The Casson invariant $\lambda_C(\mathcal{M})$ for the Brieskorn homology spheres $\mathcal{M}=\Sigma(p_1,p_2,p_3)$ is given by [4, 7] (also see, *e.g.*, Ref. 41)

$$\begin{aligned} \lambda_C\big(\Sigma(p_1,p_2,p_3)\big) = & -\frac{1}{2}\left(s(p_1\,p_2,p_3)+s(p_2\,p_3,p_1)+s(p_1\,p_3,p_2)\right) \\ & -\frac{P}{24}\left(1-\frac{1}{{p_1}^2}-\frac{1}{{p_2}^2}-\frac{1}{{p_3}^2}\right)+\frac{1}{24\,P}-\frac{1}{8} \end{aligned} \tag{4.12}$$

With this result, we have the following theorem.

**Theorem 6.** *The Casson invariant is a minus half of the number of the Eichler integrals which do not vanish at $\tau\to N\in\mathbb{Z}$;*

$$\lambda_C\big(\Sigma(p_1,p_2,p_3)\big)=-\frac{1}{2}\,\gamma(p_1,p_2,p_3) \tag{4.13}$$

The fundamental group of the Brieskorn homology spheres has the presentation (1.4), and the Casson invariant is related to the representation space of the fundamental group [4]. That is to say, we see that $\gamma(p_1, p_2, p_3)$ coincides with the cardinality of the SU(2) representation space of $\pi_1\big(\Sigma(p_1, p_2, p_3)\big)$, and that the triple $(\ell_1, \ell_2, \ell_3)$ has a one-to-one correspondence with the irreducible representation. Explicitly when we have an irreducible representation $\alpha$ of $\pi_1\big(\Sigma(p_1, p_2, p_3)\big)$, the conjugacy class of $\alpha(x_k)$ is

$$\begin{pmatrix} \mathrm{e}^{\frac{p_k - \ell_k}{p_k}\pi\mathrm{i}} & \\ & \mathrm{e}^{-\frac{p_k - \ell_k}{p_k}\pi\mathrm{i}} \end{pmatrix} \tag{4.14}$$

where the triple of integers $(\ell_1, \ell_2, \ell_3)$ satisfies a condition (4.10).

Using this correspondence with the irreducible representation of the fundamental group, we can read off the Chern–Simons invariant of the Brieskorn homology spheres [4] (see also Ref. 17);

$$\mathrm{CS}(A) = -\frac{P}{4}\left(1 + \sum_{j=1}^{3} \frac{\ell_j}{p_j}\right)^2 \mod 1 \tag{4.15}$$

**Theorem 7.** *The Chern–Simons invariant* (4.15) *for the Brieskorn homology spheres* $\Sigma(p_1, p_2, p_3)$ *is related to a limit value of the Eichler integral which can be regarded as the* $\mathbf{T}$*-matrix of the modular group; for* $N \in \mathbb{Z}$ *we have*

$$\begin{aligned} \widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(N) &= -2\,\left(\mathbf{T}^{\ell_1,\ell_2,\ell_3}\right)^N \\ &= -2\,\mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{CS}(A) N} \end{aligned} \tag{4.16}$$

*where the triple of integers* $(\ell_1, \ell_2, \ell_3)$ *satisfies eq.* (4.10)*. The triple* $(\ell_1, \ell_2, \ell_3)$ *also gives a* $SU(2)$ *representation* $\alpha$ *of the fundamental group* (1.4)*, where the conjugacy class of* $\alpha(x_k)$ *is as in eq.* (4.14).

We should point out that, with $s$ and $t$ being coprime positive integers. the number, $(s-1)\,(t-1)/2$, of integral lattice points in the 2-dimensional space has appeared as the number of the irreducible highest weight representation of the Virasoro algebra of the minimal model $\mathcal{M}(s,t)$ in the conformal field theory [3]. The character of the minimal model $\mathcal{M}(s,t)$ is modular with weight $1/2$, and it was shown [11, 12] that the Eichler integral thereof is a specific value of the colored Jones polynomial for the torus knot $\mathcal{T}_{s,t}$. Also we may say that a one-dimensional analogue of the number of the integral lattice points is realized in a torus link; the colored Jones polynomial for the torus link $\mathcal{T}_{2,2N}$ coincides with the Eichler integral of the $\widehat{su}(2)_{N-2}$ character [10], which is a $(N-1)$-dimensional representation (2.6) of the modular group with weight $3/2$. In this sense, the Brieskorn homology spheres may be regarded as a generalization of the torus knot and link from the viewpoint of the modular form. This may indicate the fact that the Brieskorn homology sphere $\Sigma(p_1, p_2, p_3)$ is homeomorphic to the $p_3$-fold cyclic branched covering of $S^3$ branched along

a torus knot $\mathcal{T}_{p_1,p_2}$ [25], and that the manifold $\Sigma(p,q,p\,q\,n\pm 1)$ can be alternatively constructed by $\pm 1/n$-surgery of the torus knot $\mathcal{T}_{p,q}$.

For our purpose to give an exact asymptotic behavior of the WRT invariant, we shall give an asymptotic expansion of the Eichler integral at $\tau \to 1/N$. The Eichler integral is no longer modular, but it has a *nearly* modular property when $\tau \in \mathbb{Q}$ as was studied in Ref. 23.

**Proposition 8.** *The Eichler integral $\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau)$ in $\tau \to 1/N$ for $N \in \mathbb{Z}_{>0}$ fulfills a nearly modular property. Namely, under the $S$-transformation, we have an asymptotic expansion in $N \to \infty$ as follows;*

$$
\begin{aligned}
\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(1/N) \simeq -\sqrt{\frac{N}{\mathrm{i}}} \sum_{\ell'_1,\ell'_2,\ell'_3} \mathbf{S}_{\ell_1,\ell_2,\ell_3}^{\ell'_1,\ell'_2,\ell'_3}\, \widetilde{\Phi}_{\boldsymbol{p}}^{(\ell'_1,\ell'_2,\ell'_3)}(-N) \\
+ \sum_{k=0}^{\infty} \frac{L(-2\,k, \chi_{2P}^{(\ell_1,\ell_2,\ell_3)})}{k!} \left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k
\end{aligned}
\tag{4.17}
$$

*Here the $L$-function is given by*

$$
L(-2\,k, \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}) = -\frac{(2\,P)^{2k}}{2\,k+1} \sum_{j=1}^{2P} \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(j)\, B_{2k+1}\left(\frac{j}{2\,P}\right) \tag{4.18}
$$

*Proof.* A proof is essentially the same with one given in Ref. 23 (see also Refs. 9–11). We use

$$
\widehat{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(z) = \frac{1}{\sqrt{2\,P\,\mathrm{i}}} \int_{z^*}^{\infty} \frac{\Phi_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau)}{\sqrt{\tau - z}}\, \mathrm{d}\tau \tag{4.19}
$$

which is defined for $z$ in the lower half plane, $z \in \mathbb{H}^-$. We mean that $z^*$ denotes a complex conjugate, and we do not have a singularity in integral. Applying the modular $S$-transformation (3.6), we see that the function $\widehat{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(z)$ has a nearly modular property;

$$
\frac{1}{\sqrt{\mathrm{i}\,z}} \sum_{\ell'_1,\ell'_2,\ell'_3} \mathbf{S}_{\ell_1,\ell_2,\ell_3}^{\ell'_1,\ell'_2,\ell'_3}\, \widehat{\Phi}_{\boldsymbol{p}}^{(\ell'_1,\ell'_2,\ell'_3)}(-1/z) + \widehat{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(z) = r_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(z;0) \tag{4.20}
$$

where we have an analogue of the period function

$$
r_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(z;\alpha) = \frac{1}{\sqrt{2\,P\,\mathrm{i}}} \int_{\alpha}^{\infty} \frac{\Phi_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\tau)}{\sqrt{\tau - z}}\, \mathrm{d}\tau \tag{4.21}
$$

for $\alpha \in \mathbb{Q}$ and $z \in \mathbb{H}^-$. On the other hand, substituting eq. (3.4) for eq. (4.19), we get for $z = x + \mathrm{i}\,y$ with $y < 0$ as follows;

$$
\begin{aligned}
\widehat{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(z) &= \frac{1}{\sqrt{2\,P\,\mathrm{i}}} \sum_{n\in\mathbb{Z}} n\, \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n) \int_{z^*}^{\infty} \frac{\mathrm{e}^{\frac{n^2}{2P}\pi\mathrm{i}\tau}}{\sqrt{\tau - z}}\, \mathrm{d}\tau \\
&= \sum_{n=0}^{\infty} \chi_{2P}^{(\ell_1,\ell_2,\ell_3)}(n)\, \mathrm{e}^{\frac{n^2}{2P}\pi\mathrm{i}z}\ \mathrm{erfc}\left(n\sqrt{-\frac{\pi\,y}{P}}\right)
\end{aligned}
$$

As a limit of $z \to \alpha \in \mathbb{Q}$, we find that

$$\widehat{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\alpha) = \widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(\alpha) \tag{4.22}$$

Then a nearly modular property (4.17) of $\widetilde{\Phi}_{\boldsymbol{p}}^{(\ell_1,\ell_2,\ell_3)}(1/N)$ follows from eq. (4.20) by setting $z \to 1/N$ and taking an asymptotic expansion of the integral (4.21) in $N \to \infty$. $\square$

It should be noted that the sum in the first term in the right hand side of eq. (4.17) runs over $D(p_1, p_2, p_3)$ distinct triples, but, as was clarified in the preceding sections, some of them vanish and there are $\gamma(p_1, p_2, p_3)$ non-zero contributions, which correspond to a sum of the flat connection (1.3).

## 5. The Witten–Reshetikhin–Turaev Invariant and the Eichler Integral

We have shown that the Eichler integral at $\tau \to N \in \mathbb{Z}$ gives the Chern–Simons invariant of the Brieskorn homology spheres $\Sigma(p_1, p_2, p_3)$. We shall show that the WRT invariant for the Brieskorn homology spheres can be expressed in terms of a limiting value of the Eichler integral at $\tau \to 1/N$ which is given in eq. (4.2).

**Theorem 9.** *For the Brieskorn homology spheres $\Sigma(p_1, p_2, p_3)$ such that*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$$

*we have*

$$\mathrm{e}^{\frac{2\pi \mathrm{i}}{N}\left(\frac{\phi(p_1,p_2,p_3)}{4} - \frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{N}} - 1\right) \tau_N\big(\Sigma(p_1, p_2, p_3)\big) = \frac{1}{2}\, \widetilde{\Phi}_{\boldsymbol{p}}^{(1,1,1)}(1/N) \tag{5.1}$$

*Proof.* Proof is essentially same with one given in Ref. 23.

We use the Gauss sum;

$$G(N) = \sum_{n=0}^{2N-1} \mathrm{e}^{-\frac{1}{2N} n^2 \pi \mathrm{i}} = \sqrt{2\,N}\, \mathrm{e}^{-\pi \mathrm{i}/4} \tag{5.2}$$

We see that

$$\begin{aligned} G(N) &= \sum_{k=0}^{2N-1} \mathrm{e}^{-\frac{1}{2N}(k-n)^2 \pi \mathrm{i}} \\ &= \mathrm{e}^{-\frac{1}{2N} n^2 \pi \mathrm{i}} \sum_{k=0}^{2N-1} \mathrm{e}^{-\frac{1}{2N} k^2 \pi \mathrm{i} + \frac{1}{N} k n \pi \mathrm{i}} \end{aligned} \tag{5.3}$$

These identities follow from the reciprocity formula (2.13).

As we see that $\widetilde{\Phi}_{\boldsymbol{p}}^{(1,1,1)}(1/N)$ is given by eq. (4.6), we have using Prop. 4 as follows;

$$\begin{aligned}\widetilde{\Phi}_{\boldsymbol{p}}^{(1,1,1)}(1/N) &= \lim_{t\searrow 0} \sum_{n=0}^{\infty} \chi_{2P}^{(1,1,1)}(n)\, \mathrm{e}^{\frac{1}{2PN} n^2 \pi \mathrm{i}}\, \mathrm{e}^{-nt} \\ &= \lim_{t\searrow 0} \frac{1}{G(P\,N)} \sum_{n=0}^{\infty} \sum_{k=0}^{2PN-1} \chi_{2P}^{(1,1,1)}(n)\, \mathrm{e}^{-\frac{1}{2PN}k^2\pi\mathrm{i} + n\left(\frac{k}{PN}\pi\mathrm{i} - t\right)}\end{aligned}$$

where in the second equality we have applied eq. (5.3).

We recall that we have the generating function of the odd periodic function $\chi_{2P}^{(1,1,1)}(n)$ as

$$\frac{\left(z^{p_1p_2} - z^{-p_1p_2}\right)\left(z^{p_2p_3} - z^{-p_2p_3}\right)\left(z^{p_1p_3} - z^{-p_1p_3}\right)}{z^{p_1p_2p_3} - z^{-p_1p_2p_3}} = \sum_{n=0}^{\infty} \chi_{2P}^{(1,1,1)}(n)\, z^n \tag{5.4}$$

Thus we get

$$\widetilde{\Phi}_{\boldsymbol{p}}^{(1,1,1)}(1/N) = \frac{1}{G(P\,N)} \sum_{\substack{k=0 \\ N\nmid k}}^{2PN-1} \mathrm{e}^{-\frac{1}{2PN}k^2\pi\mathrm{i}}\, \frac{\prod_{j=1}^{3}\left(\mathrm{e}^{\frac{k}{Np_j}\pi\mathrm{i}} - \mathrm{e}^{-\frac{k}{Np_j}\pi\mathrm{i}}\right)}{\mathrm{e}^{\frac{k}{N}\pi\mathrm{i}} - \mathrm{e}^{-\frac{k}{N}\pi\mathrm{i}}}$$

Here a contribution from the sum over $N \mid k$ is written by use of eq. (2.13) as

$$\begin{aligned}&\lim_{t\searrow 0} \frac{1}{G(P\,N)} \sum_{n=0}^{\infty} \sum_{m \mod 2P} \chi_{2P}^{(1,1,1)}(n)\, \mathrm{e}^{-\frac{N}{2P}m^2\pi\mathrm{i} + \frac{mn}{P}\pi\mathrm{i} - nt} \\ &= \lim_{t\searrow 0} \frac{1}{N} \sum_{n=0}^{\infty} \sum_{k \mod N} \chi_{2P}^{(1,1,1)}(n)\, \mathrm{e}^{\frac{2P}{N}\pi\mathrm{i}\left(k+\frac{n}{2P}\right)^2 - nt}\end{aligned}$$

which vanishes due to the odd periodicity $\chi_{2P}^{(1,1,1)}(n) = -\chi_{2P}^{(1,1,1)}(2\,P - n)$. Comparing with an expression (2.11), this completes a proof of eq. (5.1). □

**Theorem 10** ( [23]). *For the Poincaré homology sphere with* $\boldsymbol{p} = (2,3,5)$*, we have*

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right) \tau_N\bigl(\Sigma(2,3,5)\bigr) = 1 + \frac{1}{2}\, \mathrm{e}^{-\frac{1}{60N}\pi\mathrm{i}} \cdot \widetilde{\Phi}_{\boldsymbol{p}}^{(1,1,1)}(1/N) \tag{5.5}$$

*Proof.* A proof is same with Theorem 9 and was done in Ref. 23. A difference follows from a fact that the odd periodic function satisfies

$$\frac{\left(z^6 - z^{-6}\right)\left(z^{10} - z^{-10}\right)\left(z^{15} - z^{-15}\right)}{z^{30} - z^{-30}} = \frac{1}{z} + z + \sum_{n=0}^{\infty} \chi_{60}^{(1,1,1)}(n)\, z^n \tag{5.6}$$

in place of eq. (5.4). □

Combining these theorems with a nearly modular property of the Eichler integral (4.17), we obtain an asymptotic expansion of the quantum invariant in $N \to \infty$.

**Corollary 11.** *For a case of* $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}<1$ *we have*

$$
\begin{aligned}
& \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}\left(\frac{\phi(p_1,p_2,p_3)}{4}-\frac{1}{2}\right)} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}-1\right) \tau_N\bigl(\Sigma(p_1,p_2,p_3)\bigr) \\
& \qquad \simeq \sqrt{\frac{N}{\mathrm{i}}} \sum_{\ell_1,\ell_2,\ell_3} \mathbf{S}_{1,1,1}^{\ell_1,\ell_2,\ell_3} \, \mathrm{e}^{-\frac{1}{2}\pi \mathrm{i} P N \left(1+\sum_j \frac{\ell_j}{p_j}\right)^2} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{L(-2\,k,\chi_{2P}^{(1,1,1)})}{k!} \left(\frac{\pi\,\mathrm{i}}{2\,P\,N}\right)^k
\end{aligned}
\tag{5.7}
$$

Here the sum in the first term runs over $\gamma(p_1,p_2,p_3)$ distinct triples, *i.e.*, the triple $(\ell_1,\ell_2,\ell_3)$ satisfies a constraint (4.10) under a symmetry (4.11), and we have used eq. (4.16).

We give some examples below:

- $\Sigma(2,3,7)$:

  The function $\Phi^{(\ell_1,\ell_2,\ell_3)}(\tau)$ spans a $D(2,3,7)=3$-dimensional space, and the independent functions are given for $(\ell_1,\ell_2,\ell_3)=(1,1,1),(1,1,2)$, and $(1,1,3)$. For these triples we have from eq. (4.3) that $\widetilde{\Phi}^{(\ell_1,\ell_2,\ell_3)}(N)=0,-2\,\mathrm{e}^{\frac{25}{84}\pi\mathrm{i}N},-2\,\mathrm{e}^{-\frac{47}{84}\pi\mathrm{i}N}$, which shows $\mathrm{CS}(A)=-\frac{25}{168},\frac{47}{168}$. Indeed we see that $(\ell_1,\ell_2,\ell_3)=(1,1,1)$ does not satisfy a condition (4.10). This fact shows $\gamma(2,3,7)=2$, and is consistent with eq. (4.13) as we have $\lambda_C\bigl(\Sigma(2,3,7)\bigr)=-1$. As a result, we have

$$
\begin{aligned}
& \mathrm{e}^{-\frac{2\pi\mathrm{i}}{N}\frac{167}{168}} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}-1\right) \tau_N\bigl(\Sigma(2,3,7)\bigr) \\
& \qquad \simeq \sqrt{\frac{N}{\mathrm{i}}}\,\frac{2}{\sqrt{7}} \left(-\sin\left(\frac{2\,\pi}{7}\right)\mathrm{e}^{-\frac{25}{84}\pi\mathrm{i}N} - \sin\left(\frac{3\,\pi}{7}\right)\mathrm{e}^{\frac{47}{84}\pi\mathrm{i}N}\right) \\
& \qquad\qquad + \frac{1}{2}\sum_{k=0}^{\infty}\frac{L(-2\,k,\chi_{84}^{(1,1,1)})}{k!}\left(\frac{\pi\,\mathrm{i}}{84\,N}\right)^k
\end{aligned}
$$

  where

| $n \mod 84$ | 1 | 13 | 29 | 41 | 43 | 55 | 71 | 83 | others |
|---|---|---|---|---|---|---|---|---|---|
| $\chi_{84}^{(1,1,1)}(n)$ | 1 | $-1$ | $-1$ | 1 | $-1$ | 1 | 1 | $-1$ | 0 |

- $\Sigma(3,4,5)$:

  There are $D(3,4,5)=6$ independent functions $\Phi^{(\ell_1,\ell_2,\ell_3)}(\tau)$; $(\ell_1,\ell_2,\ell_3)=(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,2,1)$, and $(1,2,2)$. For these triples, we have the Eichler integral $\widetilde{\Phi}^{(\ell_1,\ell_2,\ell_3)}(N)=0,0,-2\,\mathrm{e}^{-\frac{119}{120}\pi\mathrm{i}N},-2\,\mathrm{e}^{\frac{49}{120}\pi\mathrm{i}N},-2\,\mathrm{e}^{\frac{1}{30}\pi\mathrm{i}N},-2\,\mathrm{e}^{-\frac{11}{30}\pi\mathrm{i}N}$ respectively, which indicates $\mathrm{CS}(A)=\frac{119}{240},-\frac{49}{240},-\frac{1}{60},\frac{11}{60}$. We can check that two triples, $(1,1,1)$ and $(1,1,2)$, do not satisfy a constraint (4.10). We thus have $\gamma(3,4,5)=4$, which

is consistent with $\lambda_C\bigl(\Sigma(3,4,5)\bigr) = -2$. Then we obtain

$$\begin{aligned}
&\mathrm{e}^{-\frac{2\pi \mathrm{i}}{N}\frac{71}{240}} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right) \tau_N\bigl(\Sigma(3,4,5)\bigr) \\
&\qquad \simeq \sqrt{\frac{N}{\mathrm{i}}}\,\frac{1}{\sqrt{5}} \Biggl( -\sin\left(\frac{\pi}{5}\right) \mathrm{e}^{\frac{119}{120}\pi\mathrm{i}N} + \sin\left(\frac{2\,\pi}{5}\right) \mathrm{e}^{-\frac{49}{120}\pi\mathrm{i}N} \\
&\qquad\qquad - \sqrt{2}\,\sin\left(\frac{2\,\pi}{5}\right) \mathrm{e}^{-\frac{1}{30}\pi\mathrm{i}N} + \sqrt{2}\,\sin\left(\frac{\pi}{5}\right) \mathrm{e}^{\frac{11}{30}\pi\mathrm{i}N} \Biggr) \\
&\qquad\qquad\qquad + \frac{1}{2}\sum_{k=0}^{\infty} \frac{L(-2\,k, \chi_{120}^{(1,1,1)})}{k!} \left(\frac{\pi\,\mathrm{i}}{120\,N}\right)^k
\end{aligned}$$

where

| $n \mod 120$ | 13 | 37 | 43 | 53 | 67 | 77 | 83 | 107 | others |
|---|---|---|---|---|---|---|---|---|---|
| $\chi_{120}^{(1,1,1)}(n)$ | 1 | $-1$ | $-1$ | $-1$ | 1 | 1 | 1 | $-1$ | 0 |

The Poincaré homology sphere $\Sigma(2,3,5)$ was studied in Ref. 23. In this case we have a $D(2,3,5) = 2$ dimensional representation of the modular group $PSL(2;\mathbb{Z})$, and independent functions $\widetilde{\Phi}^{(\ell_1,\ell_2,\ell_3)}(\tau)$ can be defined for $(\ell_1,\ell_2,\ell_3) = (1,1,1)$ and $(1,1,2)$. We can check that both triples fulfill a condition (4.10), and that we have $\widetilde{\Phi}^{(\ell_1,\ell_2,\ell_3)}(N) = -2\,\mathrm{e}^{\frac{1}{60}\pi\mathrm{i}N}, -2\,\mathrm{e}^{\frac{49}{60}\pi\mathrm{i}N}$, which indicates that the Chern–Simons invariant is given by $-\frac{1}{120}$ and $-\frac{49}{120}$ respectively. As we have $\phi(2,3,5) = \frac{181}{30}$, we obtain an exact asymptotic expansion as follows;

**Corollary 12** ( [23]**).**

$$\begin{aligned}
&\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\frac{121}{120}} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}} - 1\right) \tau_N\bigl(\Sigma(2,3,5)\bigr) \\
&\qquad \simeq \sqrt{\frac{N}{\mathrm{i}}}\,\frac{2}{\sqrt{5}} \left( \sin\Bigl(\frac{\pi}{5}\Bigr) \mathrm{e}^{-\frac{1}{60}\pi\mathrm{i}N} + \sin\Bigl(\frac{2\,\pi}{5}\Bigr) \mathrm{e}^{-\frac{49}{60}\pi\mathrm{i}N} \right) \\
&\qquad\qquad + \mathrm{e}^{\frac{\pi\mathrm{i}}{60N}} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{L(-2\,k, \chi_{60}^{(1,1,1)})}{k!} \left(\frac{\pi\,\mathrm{i}}{60\,N}\right)^k
\end{aligned} \tag{5.8}$$

*where*

| $n \mod 60$ | 1 | 11 | 19 | 29 | 31 | 41 | 49 | 59 | *others* |
|---|---|---|---|---|---|---|---|---|---|
| $\chi_{60}^{(1,1,1)}(n)$ | $-1$ | $-1$ | $-1$ | $-1$ | 1 | 1 | 1 | 1 | 0 |

## 6. S-matrix and the Reidemeister Torsion

An asymptotic behavior of the Witten invariant $Z_k(\mathcal{M})$ can be given from the definition (2.1). In a large $N$ limit, the first term in eqs. (5.7) and (5.8) dominate an asymptotic behavior of the quantum invariant, which will be shown to denote a contribution from flat connections. We obtain the following;

**Corollary 13.** *We have an asymptotic behavior of the Witten invariant* (1.1) *for the Brieskorn homology spheres in $N \to \infty$ as*

$$Z_{N-2}\big(\Sigma(p_1,p_2,p_3)\big) \sim \frac{1}{2}\,\mathrm{e}^{-\frac{3}{4}\pi\mathrm{i}}\,\mathrm{e}^{-\frac{\phi(p_1,p_2,p_3)}{2N}\pi\mathrm{i}} \sum_{\ell_1,\ell_2,\ell_3}\left(\sqrt{2}\,\mathbf{S}_{1,1,1}^{\ell_1,\ell_2,\ell_3}\right)\mathrm{e}^{-\frac{P}{2}\left(1+\sum_{j=1}^{3}\frac{\ell_j}{p_j}\right)^2\pi\mathrm{i}N} \tag{6.1}$$

*where the* $\mathbf{S}$*-matrix is defined in eq.* (3.6)*, and the sum runs over* $\gamma(p_1,p_2,p_3)$ *distinct triples satisfying eq.* (4.10).

We see that, except a decaying factor $\mathrm{e}^{-\frac{\phi(p_1,p_2,p_3)}{2N}\pi\mathrm{i}}$, our result proves eq. (1.3) exactly as we have seen that a dominating exponential factor denotes the Chern–Simons invariant (4.15) for the Brieskorn homology spheres.

In fact we can establish a relationship among the $\mathbf{S}$-matrix, the Reidemeister torsion, and the spectral flow;

**Theorem 14.**

$$\sqrt{2}\,\mathbf{S}_{1,1,1}^{\ell_1,\ell_2,\ell_3} = \sqrt{T_\alpha}\,\mathrm{e}^{-2\pi\mathrm{i}I_\alpha/4} \tag{6.2}$$

*where the triple* $(\ell_1,\ell_2,\ell_3)$ *satisfies eq.* (4.10).

*Proof.* A proof of the absolute value of $\sqrt{2}\,\mathbf{S}_{1,1,1}^{\ell_1,\ell_2,\ell_3}$ is straightforward since it is known [5] that the Reidemeister torsion of the Brieskorn homology sphere is given by

$$\sqrt{T_\alpha} = \frac{8}{\sqrt{P}}\prod_{j=1}^{3}\left|\sin\Big(P\,\frac{\ell_j}{p_j^{\,2}}\Big)\right| \tag{6.3}$$

To prove a part of the phase factor, we recall that the spectral flow of the Brieskorn homology spheres is given by [4]

$$I_\alpha = -3 - \left(\frac{2\left(e(\ell_1,\ell_2,\ell_3)\right)^2}{P} + \sum_{j=1}^{3}\frac{2}{p_j}\sum_{k=1}^{p_j-1}\cot\Big(\frac{k\,P\,\pi}{p_j^{\,2}}\Big)\,\cot\Big(\frac{k\,\pi}{p_j}\Big)\,\sin^2\Big(\frac{k\,e(\ell_1,\ell_2,\ell_3)\,\pi}{p_j}\Big)\right) \mod 8 \tag{6.4}$$

where

$$e \equiv e(\ell_1,\ell_2,\ell_3) = P\sum_{j=1}^{3}\frac{p_j-\ell_j}{p_j} \tag{6.5}$$

Using this we have

$$
\begin{aligned}
\mathrm{e}^{-2\pi \mathrm{i} I_\alpha/4} &= \mathrm{e}^{\frac{\pi \mathrm{i}}{2}\left(3+2\frac{e^2}{P}\right)} \prod_{j=1}^{3} \exp\left(\frac{\pi \mathrm{i}}{2}\cdot\frac{2}{p_j}\sum_{k=1}^{p_j-1}\cot\left(\frac{k\,P\,\pi}{p_j^{\,2}}\right)\cot\left(\frac{k\,\pi}{p_j}\right)\sin^2\left(\frac{k\,e\,\pi}{p_j}\right)\right)\\
&= \mathrm{e}^{\pi \mathrm{i} e}\prod_{j=1}^{3}\operatorname{sign}\left(\sin\left(\frac{q_j\,e\,\pi}{p_j}\right)\,\sin\left(\frac{e\,\pi}{p_j}\right)\right)
\end{aligned}
$$

Here in the second equality we have used $\frac{P}{p_j}\cdot q_j = 1 \pmod{p_j}$, and an identity [13, 34],

$$
\begin{aligned}
&- \mathrm{i}\ \operatorname{sign}\left(\sin\left(\frac{r\,n\,\pi}{p}\right)\,\sin\left(\frac{n\,\pi}{p}\right)\right)\,\mathrm{e}^{\pi \mathrm{i} n}\\
&\qquad = \exp\left(-\frac{\pi \mathrm{i}}{2}\left(-2+\frac{2\,r\,n^2}{p}+\frac{2}{p}\sum_{k=1}^{p-1}\cot\left(\frac{k\,q\,\pi}{p}\right)\,\cot\left(\frac{k\,\pi}{p}\right)\,\sin^2\left(\frac{k\,n\,\pi}{p}\right)\right)\right) \qquad (6.6)
\end{aligned}
$$

where we suppose $n\in\mathbb{Z}$ and $q\,r = 1 \pmod{p}$. We further see that

$$
\prod_{j=1}^{3}\sin\left(\frac{e\,\pi}{p_j}\right) = (-1)^{1+\sum_{j<k}(p_j p_k+\ell_j p_k+p_j\ell_k)}\prod_{j=1}^{3}\sin\left(\pi\,P\,\frac{\ell_j}{p_j^{\,2}}\right)
$$

and

$$
\prod_{j=1}^{3}\operatorname{sign}\left(\sin\left(\frac{q_j\,e\,\pi}{p_j}\right)\right) = 1
$$

which follows from

$$
\begin{aligned}
\sin\left(\frac{q_1\,e\,\pi}{p_1}\right) &= (-1)^{1+q_1(p_2p_3+\ell_2p_3+p_2\ell_3)}\sin\left(\pi\,\frac{\ell_1}{p_1}\cdot q_1\,\frac{P}{p_1}\right)\\
&= (-1)^{1+q_1(p_2p_3+\ell_2p_3+p_2\ell_3)-\ell_1(p_2q_3+q_2p_3)}\,\sin\left(\frac{\ell_1\,\pi}{p_1}\right)
\end{aligned}
$$

Collecting these results, we obtain

$$
\mathrm{e}^{-2\pi \mathrm{i} I_\alpha/4} = (-1)^{1+P+P\sum_{j=1}^{3}\frac{\ell_j}{p_j}+\sum_{j<k}(p_jp_k+\ell_jp_k+p_j\ell_k)}\prod_{j=1}^{3}\operatorname{sign}\left(\sin\left(\frac{P\,\ell_j\,\pi}{p_j^{\,2}}\right)\right)
$$

which proves a phase factor of eq. (6.2). □

We reconsider some examples from section 5;

- the Poincaré homology sphere $\Sigma(2,3,5)$:

  We have two triples $(\ell_1,\ell_2,\ell_3) = (1,1,1)$ and $(1,1,2)$. These respectively give the spectral flow (6.4) $I_\alpha = 4$ and $0$, which supports eq. (5.8).
- $\Sigma(2,3,7)$:

  We have $(\ell_1,\ell_2,\ell_3) = (1,1,2)$ and $(1,1,3)$, which give $I_\alpha = 6$ and $2$ respectively. This is consistent with a result in previous section.

- $\Sigma(3,4,5)$:

  We have four triples $(\ell_1,\ell_2,\ell_3)=(1,1,3),(1,1,4),(1,2,1)$, and $(1,2,2)$. For these irreducible representations, we get from eq. (6.4) $I_\alpha = 2,4,6$, and $0$. This result is consistent with an asymptotic expansion given before.

## 7. THE OHTSUKI INVARIANT

From asymptotic expansions (5.7) and (5.8) of the WRT invariant, we can introduce a formal power series which is ignored in section 6. This series may denote a trivial connection contribution [22, 38, 39]. By regarding $\mathrm{e}^{2\pi\mathrm{i}/N}$ as $q$, we can define $\tau_\infty(\mathcal{M})$ for a case of $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}<1$

$$q^{\frac{\phi(p_1,p_2,p_3)}{4}-\frac{1}{2}}\,(q-1)\,\tau_\infty\big(\Sigma(p_1,p_2,p_3)\big)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{L(-2\,k,\chi_{2P}^{(1,1,1)})}{k!}\left(\frac{\log q}{4\,P}\right)^k \tag{7.1}$$

and for the Poincaré homology sphere

$$q^{\frac{121}{120}}\,(q-1)\,\tau_\infty\big(\Sigma(2,3,5)\big)=q^{\frac{1}{120}}+\frac{1}{2}\sum_{k=0}^{\infty}\frac{L(-2\,k,\chi_{60}^{(1,1,1)})}{k!}\left(\frac{\log q}{120}\right)^k \tag{7.2}$$

With these definitions the Ohtsuki invariant [30] is defined by the formal series for $\tau_\infty(\mathcal{M})$;

$$\tau_\infty\big(\Sigma(p_1,p_2,p_3)\big)=\sum_{n=0}^{\infty}\lambda_n\big(\Sigma(p_1,p_2,p_3)\big)\cdot(q-1)^n \tag{7.3}$$

Infinite series in eqs. (7.1) and (7.2) originate from an asymptotic expansion of the integral $r_{\boldsymbol{p}}^{(1,1,1)}(1/N;0)$ defined in eq. (4.21), which appears as a *tail* of the nearly modular property of the Eichler integral. It is noted that an integral formula for the Ohtsuki invariant was studied in Refs. 21, 39 by a different method.

To compute an explicit form of $\lambda_n(\mathcal{M})$, we use the Stirling number of the first kind defined by

$$\prod_{j=0}^{n-1}(x-j)=\sum_{m=0}^{n}S_n^{(m)}\,x^m \tag{7.4}$$

As the Stirling number satisfies (see, *e.g.*, Ref. 1)

$$\frac{(\log q)^m}{m!}=\sum_{n=m}^{\infty}S_n^{(m)}\,\frac{(q-1)^n}{n!} \tag{7.5}$$

we can easily obtain the following expression.

**Theorem 15.** *Let the function $\Lambda_n(p_1,p_2,p_3)$ be defined by*

$$\Lambda_n(p_1,p_2,p_3) = \frac{1}{2}\,\frac{1}{(n+1)!}\sum_{m=1}^{n+1} S_{n+1}^{(m)} \left(\frac{2-\phi(p_1,p_2,p_3)}{4}\right)^m \times \sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{P\,(2-\phi(p_1,p_2,p_3))}\right)^k L(-2\,k,\chi_{2P}^{(1,1,1)}) \tag{7.6}$$

*Then the invariant $\lambda_n(\mathcal{M})$ is computed as follows;*

$$\lambda_n\big(\Sigma(p_1,p_2,p_3)\big) = \begin{cases} \Lambda_n(p_1,p_2,p_3) & \text{for } \dfrac{1}{p_1}+\dfrac{1}{p_2}+\dfrac{1}{p_3}<1 \\ \Lambda_n(2,3,5)+(-1)^{n+1} & \text{for the Poincaré homology sphere} \end{cases} \tag{7.7}$$

We note that a value of the $L$-function, which is given by eq. (4.18), can be computed easily from a generating function for a case of $(\ell_1,\ell_2,\ell_3)=(1,1,1)$;

$$\begin{aligned} -2\,\mathrm{ch}(z) + 2\,\frac{\mathrm{sh}(6\,z)\;\mathrm{sh}(10\,z)}{\mathrm{ch}(15\,z)} &= -2\,\frac{\mathrm{ch}(9\,z)\;\mathrm{ch}(5\,z)}{\mathrm{ch}(15\,z)} \\ &= \sum_{n=0}^{\infty}\frac{L(-2\,n,\chi_{60}^{(1,1,1)})}{(2\,n)!}\,z^{2n} \end{aligned} \tag{7.8}$$

and in a case of $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}<1$ we have

$$4\,\frac{\mathrm{sh}(p_1\,p_2\,z)\;\mathrm{sh}(p_1\,p_3\,z)\;\mathrm{sh}(p_2\,p_3\,z)}{\mathrm{sh}(p_1\,p_2\,p_3\,z)} = \sum_{n=0}^{\infty}\frac{L(-2\,n,\chi_{2P}^{(1,1,1)})}{(2\,n)!}\,z^{2n} \tag{7.9}$$

See Table 1 for explicit values of $\lambda_n(\mathcal{M})$ for some Brieskorn spheres.

We can rewrite eq. (7.7) for the first three terms as follows;

$$\lambda_0\big(\Sigma(p_1,p_2,p_3)\big) = 1 \tag{7.10}$$

$$\lambda_1\big(\Sigma(p_1,p_2,p_3)\big) = -\frac{1}{4}\left(\phi + P\left(1-\frac{1}{{p_1}^2}-\frac{1}{{p_2}^2}-\frac{1}{{p_3}^2}\right)\right) \tag{7.11}$$

$$\lambda_2\big(\Sigma(p_1,p_2,p_3)\big) = \frac{1}{12}\left(\frac{3\,\phi^2+12\,\phi-4}{8} + \frac{3\,P}{4}\,(\phi+2)\left(1-\sum_{j=1}^{3}\frac{1}{{p_j}^2}\right) + \frac{P^2}{8}\left(2\left(1-\sum_{j=1}^{3}\frac{1}{{p_j}^4}\right)+5\left(1-\sum_{j=1}^{3}\frac{1}{{p_j}^2}\right)^2\right)\right) \tag{7.12}$$

where we mean $\phi=\phi(p_1,p_2,p_3)$ defined in eq. (2.12). As was proved in Refs. 27, 28 we see that

$$\lambda_0\big(\Sigma(p_1,p_2,p_3)\big) = 1 \tag{7.13}$$

$$\lambda_1\big(\Sigma(p_1,p_2,p_3)\big) = 6\,\lambda_C\big(\Sigma(p_1,p_2,p_3)\big) \tag{7.14}$$

| $\mathcal{M}$ | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_6$ | $\lambda_7$ | $\lambda_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $\Sigma(2,3,5)$ | 1 | −6 | 45 | −464 | 6224 | −102816 | 2015237 | −45679349 | 1175123730 |
| $\Sigma(2,3,7)$ | 1 | −6 | 69 | −1064 | 20770 | −492052 | 13724452 | −440706098 | 16015171303 |
| $\Sigma(2,3,11)$ | 1 | −12 | 198 | −4564 | 136135 | −4979568 | 215636785 | −10785653847 | 611802510704 |
| $\Sigma(2,3,13)$ | 1 | −12 | 246 | −6916 | 248171 | −10848488 | 559466999 | −33256127501 | 2238888918356 |
| $\Sigma(2,3,17)$ | 1 | −18 | 459 | −16404 | 757689 | −42883758 | 2872307319 | −222158381412 | 19483805436567 |
| $\Sigma(2,3,19)$ | 1 | −18 | 531 | −21660 | 1131375 | −72097914 | 5424649644 | −470672677647 | 46266101270760 |
| $\Sigma(2,5,7)$ | 1 | −12 | 222 | −5596 | 179985 | −7054432 | 326278974 | −17397305298 | 1050720467092 |
| $\Sigma(2,5,9)$ | 1 | −18 | 411 | −12900 | 523445 | −26063974 | 1537243785 | −104755839122 | 8097415424747 |
| $\Sigma(2,5,11)$ | 1 | −18 | 531 | −21180 | 1074975 | −66390674 | 4834590264 | −405657513711 | 38541528405358 |
| $\Sigma(2,7,13)$ | 1 | −36 | 1674 | −106884 | 8799855 | −887883368 | 106042462590 | −14627548503126 | 2288188525438231 |
| $\Sigma(2,7,15)$ | 1 | −36 | 2010 | −152244 | 14703739 | −1730017752 | 240158450652 | −38429255864768 | 6964744996791857 |
| $\Sigma(2,11,17)$ | 1 | −72 | 6948 | −918744 | 156141090 | −32466056280 | 7983578235864 | −2266232536578132 | 729278178446689719 |
| $\Sigma(2,11,19)$ | 1 | −78 | 8481 | −1261160 | 240437790 | −56007699396 | 15418103295783 | −4897366295772501 | 1763003584636961535 |
| $\Sigma(2,11,21)$ | 1 | −90 | 10695 | −1742580 | 365286685 | −93734684478 | 28454533630530 | −9972833783229875 | 3962937841176563555 |
| $\Sigma(3,4,5)$ | 1 | −12 | 198 | −4324 | 119455 | −4012828 | 159008935 | −7263759799 | 375878922067 |
| $\Sigma(3,4,7)$ | 1 | −18 | 411 | −12420 | 476645 | −22300734 | 1232660885 | −78624211186 | 5684458291305 |
| $\Sigma(3,4,11)$ | 1 | −30 | 1065 | −49960 | 2988770 | −218577416 | 18915594545 | −1890831231245 | 214380047624390 |
| $\Sigma(3,4,13)$ | 1 | −30 | 1305 | −74680 | 5390950 | −472424616 | 48786184083 | −5804487459615 | 781992244219680 |
| $\Sigma(3,4,23)$ | 1 | −60 | 4470 | −441740 | 55544845 | −8523297832 | 1546017437658 | −323709524734970 | 76844211062714480 |
| $\Sigma(3,4,25)$ | 1 | −60 | 4950 | −540140 | 74583605 | −12524972472 | 2481232929734 | −566688382409942 | 146614680355157664 |
| $\Sigma(3,5,7)$ | 1 | −24 | 684 | −25640 | 1222766 | −71219336 | 4906476652 | −390356879176 | 35220329064877 |
| $\Sigma(3,5,8)$ | 1 | −24 | 804 | −35360 | 1961016 | −132053796 | 10481576931 | −958711945083 | 99307064129868 |
| $\Sigma(3,5,11)$ | 1 | −36 | 1626 | −96756 | 7300091 | −671133288 | 72881361140 | −9132371657296 | 1297049789194653 |
| $\Sigma(3,5,13)$ | 1 | −42 | 2247 | −158396 | 14146195 | −1538635378 | 197615046741 | −29280417770120 | 4916816674605230 |
| $\Sigma(4,5,7)$ | 1 | −30 | 1185 | −60880 | 3934980 | −308750176 | 28560576934 | −3045183200982 | 367764306523118 |
| $\Sigma(4,7,9)$ | 1 | −60 | 4230 | −385820 | 44419865 | −6227423152 | 1031445122700 | −197181302142540 | 42735797918044660 |

Table 1: $\lambda_n\big(\Sigma(p_1,p_2,p_3)\big)$

where $\lambda_C(\mathcal{M})$ is the Casson invariant (4.12) for $\mathcal{M}$. It is noted that $\lambda_2\big(\Sigma(p_1, p_2, p_3)\big)$ is calculated in Ref. 40.

## 8. DISCUSSIONS

We have studied the Witten–Reshetikhin–Turaev invariant for the Brieskorn homology spheres (2.11) by use of properties of the Eichler integral of the half-integral weight based on the method of Ref. 23. The WRT invariant coincides with a limiting value of the Eichler integral at $\tau \to 1/N$ for $N \in \mathbb{Z}$, and the nearly modular property (4.17) of the Eichler integral gives an exact asymptotic behavior of the WRT invariant (5.7) and (5.8). With the correspondence between the modular form and the quantum invariant, we can give an interpretation for the invariants of manifold such as the Chern–Simons invariant, the Ohtsuki invariant, the Casson invariant, Reidemeister torsion, and the spectral flow from a point of view of the modular form. Especially the number of the non-vanishing Eichler integrals at $\tau \to N \in \mathbb{Z}$ is related to the Casson invariant, and there exists a correspondence with an irreducible SU(2) representation of the fundamental group. In our previous papers [10–12], we revealed a relationship between a specific value of the colored Jones polynomial for the torus knot and link, $\mathcal{T}_{s,t}$ and $\mathcal{T}_{2,2m}$, and the Eichler integral of the half-integral weight modular form. Therein shown was that an exact asymptotic behavior has a form of eqs. (5.7) and (5.8), and that a generating function of a tail polynomial part, or the Ohtsuki invariant, is an inverse of the Alexander polynomial $\frac{A^{1/2}-A^{-1/2}}{\Delta(A)}$. So we may conclude that the left hand side of eq. (7.9) plays a role of the inverse of the Alexander polynomial. In the same manner, we may define an analogue of the Casson invariant by a minus half of the number of the non-vanishing Eichler integrals at $\tau \in \mathbb{Z}$. We collect these correspondence in the SU(2) quantum invariants in Table 2.

| | torus link $\mathcal{T}_{2,2m}$ | torus knot $\mathcal{T}_{s,t}$ | $\Sigma(p_1, p_2, p_3)$ |
|---|---|---|---|
| $d$ | $1$ | $2$ | $3$ |
| "Casson" $\lambda_C$ | $-\dfrac{1}{2}\,(m-1)$ | $-\dfrac{1}{4}\,(s-1)\,(t-1)$ | $-\dfrac{1}{2}\,\gamma(p_1, p_2, p_3)$ |
| "Alexander" $\Delta(A)$ | $\dfrac{A^m - A^{-m}}{A^{\frac{1}{2}} + A^{-\frac{1}{2}}}$ | $\dfrac{(A^{\frac{st}{2}} - A^{-\frac{st}{2}})\,(A^{\frac{1}{2}} - A^{-\frac{1}{2}})}{(A^{\frac{s}{2}} - A^{-\frac{s}{2}})\,(A^{\frac{t}{2}} - A^{-\frac{t}{2}})}$ | $\dfrac{(A^{\frac{1}{2}} - A^{-\frac{1}{2}})^2}{\prod_{j=1}^{3}(A^{\frac{1}{2p_j}} - A^{-\frac{1}{2p_j}})}$ |

Table 2: We give an interpretation for the "Casson invariant" and the "Alexander polynomial" from the view point of the modular forms. "Dimension" $d$ denotes that the number of the modular form is related to the number of the integral lattice point in $d$-dimensional space.

In Ref. 23 discussed also is a relationship between the WRT invariant for the Poincaré homology sphere and the Ramanujan mock theta function. We hope to report on the $q$-series identity associated with the quantum invariant for the Brieskorn homology sphere using a surgery description with an expression of the colored Jones polynomial for the torus knot given in Ref. 8 (see also Ref. 24).

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